It will be understood from this problem that many similar problems in gunnery can be solved by the aid of this diagram with an accuracy probably near enough for most practical purposes.

An accurately drawn ballistic diagram accompanies this paper and is reproduced in the folded plate, reduced to slightly less than half size. It may be used in the way exemplified by fig. 4 for the solution of problems of gunnery relating to direct-angle fire. The scales to which the original of this diagram is drawn are:—

Velocity, 1 inch = 200 feet per second.
Distance, 1 inch = 1000 feet.
Time, 1 inch = 1 second.
Energy, 1 inch = 5 foot-tons.

Auxiliary curves:—

$1/pg$, 1 inch = 0·002 unit.
g/v, 1 inch = 0·005 unit.

Surface Friction: Experiments with Steam and Water in Pipes.

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During the past hundred years much work, both theoretical and experimental, has been carried out with a view to determining the character of the laws governing the resistance to tangential motion between solid surfaces and liquids or gases. A general relation between the dimensions of the surface, the velocity, the density of the fluid, and its viscosity had been surmised as a consequence of the laws of motion by Stokes, Helmholtz, and Osborne Reynolds, but it was left to Lord Rayleigh* to show, from the principle of dynamical similarity, that the phenomena involved could be expressed definitely by a simple mathematical formula.

The laws governing the friction between solid surfaces and water have formed the subject of experimental investigations by Froude, Osborne Reynolds, Darcy, etc., whilst the parallel case of the resistance to motion between solid surfaces and perfect gases has occupied the attention of Zahm, Brix, Stockalper, and others. Practically all these investigators devoted their energies to experimental determinations of the friction in the medium which they employed, and it was not until the subject was taken

* 'Phil. Mag.,' p. 321 (1899). 'Advisory Committee for Aeronautics Report,' 1909–10, p. 38.

up by Stanton and Pannell* that any attempt was made to investigate the similarity, under certain conditions, of the motion of fluids which differed widely amongst themselves in their properties of densities and viscosities. The same investigators also took up the question of the limits of accuracy of the formulæ currently accepted at the time and used in calculations of surface friction.

Experimental work upon the friction between steam and metal or other surfaces has been carried out by Barry† and Carpenter,‡ the former on long steam pipes in the Lambrecht Mine at Anzin, the latter in the laboratories of the Sibley College. Barry's results have been reduced to an approximate formula by Ledoux.

These investigations extended only over a relatively small range of velocity, and the accuracy of the results obtained was insufficient to establish any systematic departure from the velocity squared law; the investigators therefore contented themselves with finding values of the constant k suitable for use at the particular mean velocity of their experiments, assuming that

$$R = kv^2.$$

The object of the present research was in the first place to obtain accurate data for the resistance to flow in pipes conveying steam and other vapours with a view to checking the accuracy with which Osborne Reynolds' dimensional law of flow could be made to represent the friction of these fluids. The work was afterwards extended to ranges over which this law did not accurately apply, and the results have been used to extend the work of Stanton and Pannell to the important case of friction between steam and metal surfaces.

Theoretical Expressions for Flow.

The expression referred to above as given by Lord Rayleigh and connecting the linear dimensions of the body with the density, the velocity and the kinematical viscosity of the fluid may be written

$$R = \rho v^2 F(vd/\nu),$$

where F is a function of the one variable (vd/ν),

R = resistance per unit area,

ρ = density of fluid,

v = velocity,

d = diameter of the pipe,

ν = kinematical viscosity of the fluid $= \mu/\rho$.

* 'Phil. Trans.,' A, vol. 214, p. 199 (1914).

† 'Annales des Mines,' Ser. 2, 1892. Ledoux on "Loss of Pressure in Pipes."

‡ 'Proc. Inst. Mech. Eng. of America,' 1893.

Previous to the establishment of the above expression Reynolds had shown from the theory of dimensions that, assuming the frictional resistance to flow to vary as the nth power of the mean velocity, the relation between the resistance, the viscosity, the density, and the velocity would be expressed by

$$\partial p = kd^{n-3}\mu^{2-n}\rho^{n-1}v^n\partial l.$$

The assumption made by Reynolds is equivalent to taking the function of vd/ν in Rayleigh's formula to be the power $n-2$ or that

$$\mathrm{R} = k\rho v^2(vd/\nu)^{n-2}.$$

In the reduction of his experimental work on the flow of water in pipes Reynolds* found that over the range of velocities used the expression held with considerable accuracy in any one pipe, but that n varied in some manner with the roughness of the surface, and also increased with the pipe diameter. Above the critical velocity n varied from 1·72 for a smooth pipe of $\frac{1}{4}$ inch diameter up to 1·92 for Darcy's rough pipes of 20 inches diameter.

Stanton and Pannell found that a considerable deviation from the simple exponential law became apparent when the velocity in the pipe varied from 0 to 3000 cm. per second, but that for small ranges Reynolds' formula fitted in with the experimental results to a fair degree of accuracy.

Limits of Present Experimental Work.

In the present research the moving fluid was in most cases either dry saturated or slightly wet steam, and the experimental range extended habitually to 4000 cm. per second, although in a few experiments a velocity of 6000 cm. per second was obtained. It was found undesirable to extend the work to speeds above this, since extremely high speeds, such as can be obtained easily with steam, are associated with superheating down the pipe, making the reduction of the results obtained too complicated and unreliable. The maximum value of vd/ν, however, amounts to about 500×10^3 C.G.S. units, since by working at a high pressure the value of the kinematical viscosity may be considerably reduced below that of air at atmospheric pressure and a corresponding increase in vd/ν effected.

For the low values of vd/ν corresponding to the experimental pipes used, water was adopted as the moving fluid, and thus the curve extended backwards to values of this function corresponding to the point where the character of the curve changes, and below which the resistance varies directly as the velocity.

In all pipes values of the resistance corresponding to low velocities of

* 'Phil. Trans.,' 1883, p. 935.

steam were obtained, although the work becomes increasingly difficult as the velocity decreases, until eventually reliable results become impossible owing to the serious effects of condensation.

Condensation is also of importance when the experimental pipe is long, since the effect of heat loss causes the steam to become wetter as it flows down the pipe, the specific volume and the density are altered, and the velocity is changed slightly from point to point. The difficulty, however, of measuring the extremely small pressure differences which obtain in short lengths of tubing caused the author to adopt pipes varying in length from about 30 to 80 feet, and correcting for the effects of condensation and change in velocity in the manner hereafter described.

For the final plotting of the results, the experimental values were reduced to those obtaining over the first unit length of the pipe, thus making a comparison possible with other investigators' results for air and water.

Experimental Apparatus.

The apparatus consisted of three mild steel steam pipes of commercial quality, the joints being made by screwed couplings overlapping the ends of the two pipes, which were butted one against the other, leaving a uniform bore across the joint. Each pipe was connected by means of a short leading-in length with a large separator supplied with steam by a 300-H.P. Babcock and Willcox boiler. The large capacity of the boiler rendered it easy to maintain steady pressure conditions during the trials, which frequently occupied upwards of one hour.

The measurements of pressure drop were made at two points in the tube a definite distance apart. At these points, A and B (fig. 1), a number of small holes (C, C) were drilled round the pipe, the inside being carefully smoothed down.

It follows from the work of Bramwell and Fage,* on Pitot tubes, that by this method the static pressure at any point may be accurately determined.

The difference in pressure existing at the two ends of the measured length was obtained for the bulk of the experiments by a U-tube containing mercury, D (fig. 1), the two legs of the U being connected by couplings E and F and tubes H and I to the rings of holes (C, C) at the two "take-off" points A and B. For the smaller pressure differences the U-tube gauge was inverted (fig. 2), water being used to measure the difference in head, an

* 'Advisory Committee for Aeronautics,' 1912–13, p. 33. Bramwell and Fage on "A Determination on the Whirling Table of the Pressure Velocity Constant for a Pitot (Velocity Head and Static Press) Tube; and on the Absolute Measurements of the Velocity in Aeronautical Work."

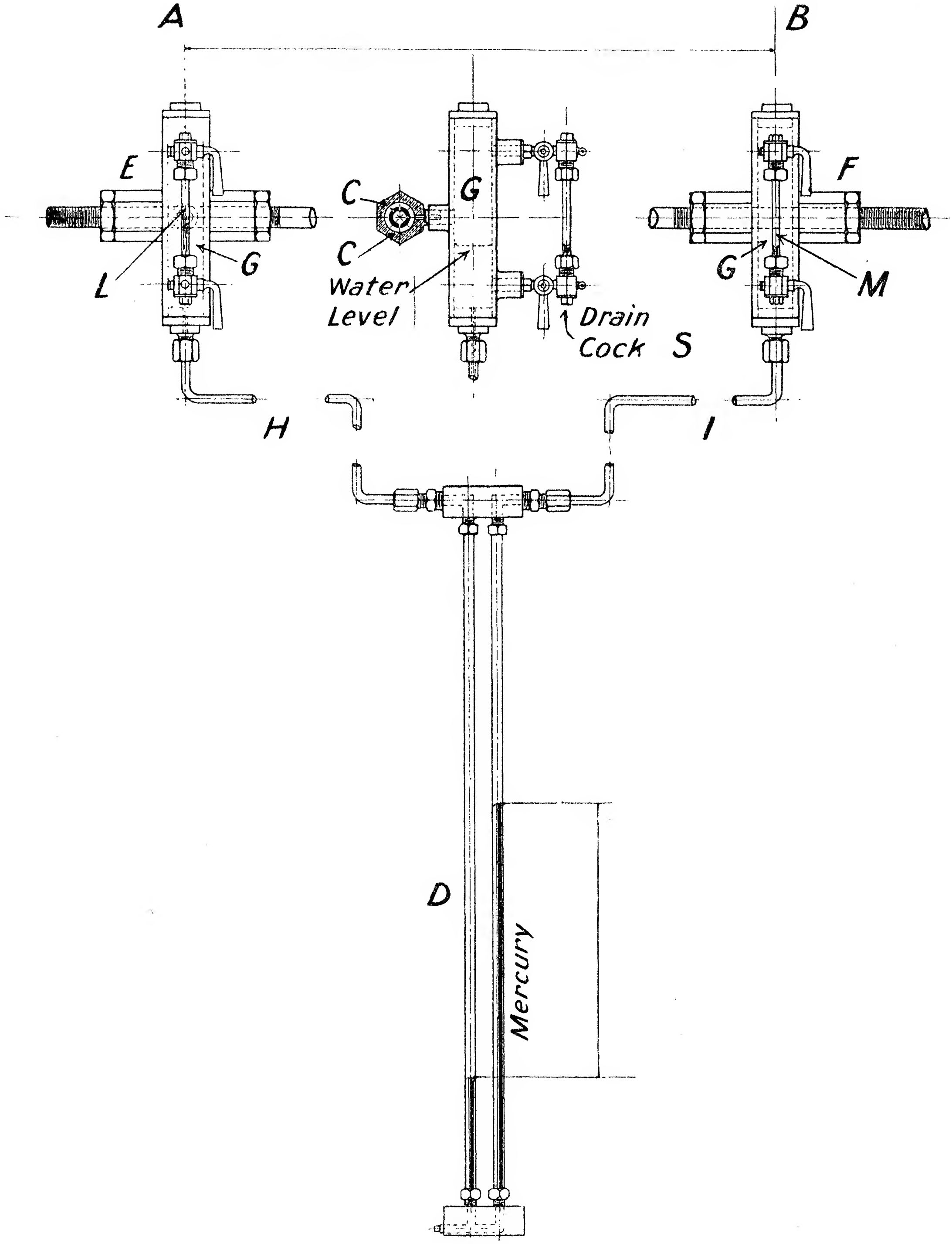

Fig. 1.

equalised air pressure above the water surfaces in the inverted U-tube being obtained by introducing a small quantity of compressed air from the compressed air vessel S into the gauge at the cross limb N of the U.

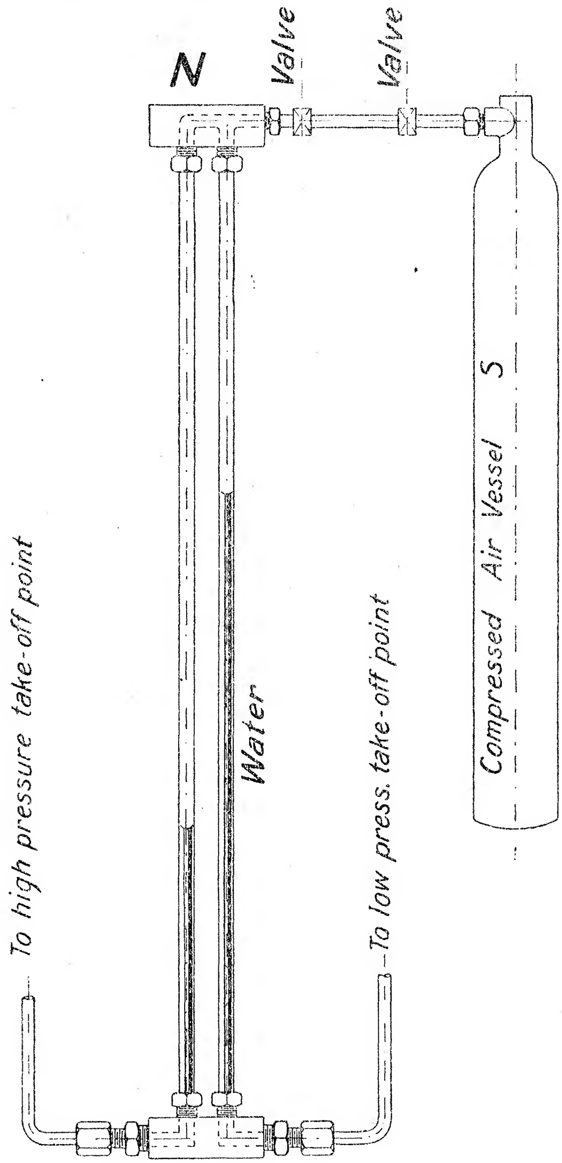

Fig. 2.

For accurate determinations of the pressure, it was necessary to know the head of water contained in the connecting tubes. This was effected by the provision of two chambers, G (see fig. 1), which were coupled between the

take-off castings E and F and the pressure tubes H and I. The level of the water in these chambers could be read on the gauge glasses L and M. A scale for the U-tube was marked off, being calculated in pounds per square inch, corresponding to mercury in water, and scales for the gauge glasses in pounds, corresponding to the water heads, the two latter being reduced to the same zero by careful levelling.

It will be seen, then, that the pressure difference at the two ends of the experimental pipe was equal to the reading in pounds of the mercury gauge added to the difference in pounds between the two readings of the water gauge glasses. Both collectors G were carefully lagged to reduce the condensation to a minimum. The amount of water condensed was collected from the gauges by the cocks S, and measured by weighing in a known quantity of water. In the upper of the two collectors the condensed water was due entirely to the heat loss from the collector itself, and may be taken as a measure of the steam flow through the pressure holes C, C. The amount was small, and such that the dynamic head corresponding to its velocity did not in any case exceed 0·08 per cent. of the least pressure difference observed. The pressure, as measured, could therefore be considered static.

In the lower of the two collectors the water collected was due both to a proportion of the water condensed in the experimental main and to that condensed in the collector. It is clear that the former will make no difference to the measurements of the static pressure, and the latter will only affect the readings to the same extent as that in the high-pressure collector. The total amount of water collected in the low-pressure collector was added to that discharged in the hot well, since all results have been reduced to the initial velocity in the pipe, and this water in the form of steam has already passed the high-pressure take-off point. It is obvious that proper precautions must be taken to ensure that water and mercury are continuous in the pressure-measuring devices, and not intercepted by air spaces.

After passing the second of the two take-off points, the steam was led into a separator, and thence passed through a plug valve to the condenser. The plug valve was arranged to act as a throttling calorimeter, a pressure gauge and a thermometer pocket being inserted in the pipe on the low-pressure side. Condensed water was pumped out of the condenser by an air pump, and collected in a calibrated tank. The separator was fitted with a gauge glass and a scale calibrated to read in pounds collected in the separator. The whole of the piping was well clothed with slag wool wrapped with canvas, the thickness of the covering averaging $1\frac{1}{4}$ inches throughout. A number of experiments were also carried out on the bare pipe.

The condition of the inlet steam was adjusted by means of a throttle valve on the inlet pipe, situated at a distance of about 6 feet from the high-pressure measuring point, the method being to obtain a few degrees of superheat at a thermometer pocket and Bourdon gauge 2 feet from the high-pressure measuring point. Dry steam at the entry of the experimental length was thus ensured. Any instability of flow due to the thermometer would die down in the 20 or more diameters before it reached the take-off holes, and in any case its effects would be negligible in view of the long experimental pipes used. Finally Bourdon pressure gauges were connected directly to the take-off muffs in order to obtain readings at high velocities.

Experimental Results.

The observations taken consisted of the initial temperature and pressure of the steam, the pressure difference between the take-off points, the dryness on entering the second separator and the weight of steam discharged, including that condensed in the lower of the two take-off muff chambers (see Columns 1, 3, 7, and 8 of Table VI). Tests were also made on the heat loss from the main by allowing an extremely small flow through the pipe, collecting at the same time the condensed water.

The method of inferring the velocity from the volume passing and the diameter of the conduit necessitates a very accurate determination of the latter since the value of $R/\rho v^2$ depends on the fifth power of the diameter. For this purpose a measured length of the pipe was filled with water, emptied and the quantity accurately weighed, the diameter being inferred from the volume. The mean result obtained from six determinations was taken as the correct value. This method appeared to the author to be the most suitable one, since by leaving a wet surface on the interior of the pipe it approximated more accurately to the actual diameter and state of the surface when steam was flowing. Condensation continually takes place and a stationary film of water is of necessity adhering to the roughness of the interior surface.

Reduction of the Experimental Values.

In order to determine the form of the law governing the frictional losses in any one pipe at a given initial steam pressure Reynolds' form of Rayleigh's law was adopted, since the range of velocity was small and well within the limits of accuracy of the exponential law. Assuming then that

$$\partial p = h\mathrm{D}^{n-3}\mu^{2-n}\rho^{n-1}v^{n}\,\partial l,$$

let λ = weight of steam condensed per second per linear foot of pipe, this being the intrinsic amount of water present due to the loss through the pipe covering partly neutralised by drying effects of friction,

w_0 = weight of dry steam entering pipe per second,

w = weight of dry steam flowing past a given point distant l along pipe;

then
$$w = w_0 - \lambda l.$$

The relation between the volume of unit weight of dry steam and its absolute pressure has for this purpose been taken as that given by $pv^{1\cdot064}$ = constant, since the change in specific volume down the pipe is small.

Density of dry vapour $= 1/v = \mathrm{K}p^{\cdot94}$.

Volume of weight w of dry vapour $= w/\mathrm{K}p^{\cdot94} = \mathrm{V} = (w_0 - \lambda l)/\mathrm{K}p^{\cdot94}$.

But the actual substance flowing past any section of the pipe is a mixture consisting of steam and a small proportion of fine drops of water.

The density of the mixture $= w_0\mathrm{K}p^{\cdot94}/w$, and therefore
$$\rho = w_0\mathrm{K}p^{\cdot94}/(w_0 - \lambda l).$$

The velocity of the fluid $= \dfrac{\mathrm{V}}{\text{area}} = \left(\dfrac{w_0 - \lambda l}{\mathrm{K}p^{\cdot94}}\right)\dfrac{4}{\pi\mathrm{D}^2}$. Inserting these values of ρ and v in the dimensional formula, we get
$$\partial p = k\mathrm{D}^{n-3}\mu^{2-n}\left(\frac{w_0\mathrm{K}p^{\cdot94}}{w_0 - \lambda l}\right)^{n-1}\left(\frac{4(w_0 - \lambda l)}{\pi\mathrm{D}^2\mathrm{K}p^{\cdot94}}\right)^n \partial l,$$
which may be written
$$p^{\cdot94}\partial p = \frac{\mathrm{K}\mu^{2-n}}{\mathrm{D}^{3+n}}\left(w_0{}^n\,\partial l - \frac{w_0{}^n\lambda l\,\partial l}{w_0}\right).$$

Assuming the viscosity to be sensibly constant down the pipe* and integrating between lengths 0 and l, pressures p_1 and p_2, we get
$$p_1{}^{1\cdot94} - p_2{}^{1\cdot94} = \frac{\mathrm{M}\mu^{2-n}w_0{}^n}{\mathrm{D}^{3+n}}\left(1 - \frac{\lambda l}{2w_0}\right)l.$$

In order to plot the results the formula was put in the form
$$\frac{p_1{}^{1\cdot94} - p_2{}^{1\cdot94}}{1 - \lambda l/2w_0} = \mathrm{A}w_0{}^n l, \qquad (1)$$
for any given value of the diameter and the initial pressure. The values of

* The justification for this assumption may be inferred from an inspection of the final plot, fig. 3. At low pressures in small pipes, *i.e.* where this error would have its greatest effect, the value of n works out at about 1·79. At pressures of 0·8 kgrm. per sq. cm. abs. the greatest observed difference of pressure was 0·2 kgrm. per sq. cm., corresponding to a theoretical change of 2 per cent. in the viscosity. Since the viscosity appears in the formula with the index $2-n$ or say 0·2, the maximum relative value of error will be $1{\cdot}02^{0\cdot2}/1$, or say 0·5 per cent.

the left hand side were then worked out from the experimental values of p_1, p_2, and the condensation down the pipe.

If now p_3 represents the pressure at a point distant unit length from p_1 we get

$$\frac{p_1^{1\cdot 94}-p_3^{1\cdot 94}}{1-\lambda/2w_0} = Aw_0^n, \tag{2}$$

but over unit length at the entry to the experimental length

$$1-\lambda/2w_0 = 1 \text{ approx.}$$

Dividing (1) by (2) we are thus led to the equation

$$p_1^{1\cdot 94}-p_3^{1\cdot 94} = \left(\frac{p_1^{1\cdot 94}-p_2^{1\cdot 94}}{1-\lambda l/2w_0}\right)1/l = 1\cdot 94\,(p_1-p_3)\,p_1^{\cdot 94},$$

since p_1-p_3 is small; from which the drop in pressure over unit length at the entrance to the pipe can be calculated.

The principal values of R the resistance per unit surface and the velocities, together with values of $R/\rho v^2$ and vd/ν, are given in Tables I, II, III, IV, and V. Details of the reduction of typical experimental values will be found in Table VI.

In the reductions, values for the viscosities* at the various steam temperatures have been obtained by linear extrapolation of the results given by Puluj, Kundt and Warburg, and Meyer and Schumann for temperatures 0°, 15° and 100° respectively, this method being justified since inspection of the resultant curve (fig. 3) shows it to be comparatively insensitive to slight inaccuracies in the values of the viscosity, thus an error in the estimation of the viscosity as great as 10 per cent. would only cause a horizontal shift of the point involved by log 1·1 = 0·04, an amount which in view of the comparatively slow change of ordinate with abscissa is negligible and commensurate with the accuracy of other parts of the investigation.

In the bulk of the experiments the pressure head corresponding to the increase in velocity did not exceed 0·5 per cent. In the cases, however, where large pressure drops were experienced a correction was applied to allow for the proportion of the pressure drop necessary to accelerate the steam from the initial to the final velocity.

* The values of the viscosities for saturated steam at temperatures above 100° C. are not accurately known. From experiments carried out below that temperature it appears, however, to follow a law $\mu = at+\beta$, a and β being approximately $0\cdot 42\times 10^{-6}$ and 90×10^{-6} respectively.

Table I.—Experiments with Water. Internal Diameter of Pipe, 1·07 cm.

v, mean velocity.	R.	$\frac{R}{\rho v^2} 10^2$.	$\frac{vd}{\nu} 10^{-3}$.	Temperature.
cm./sec.	dynes/cm.^{2}			
48·7	12·6	0·53	4·56	15° C.
66·2	21·6	0·49	6·23	
86·6	35·2	0·47	8·13	
101·4	45·4	0·44	9·52	
132·0	76·2	0·44	12·4	
151·0	91·9	0·40	14·2	
181·0	126·0	0·38	17·0	
220·0	184·0	0·38	20·6	
281·0	277·0	0·35	26·4	
352·0	431·0	0·35	33·0	

Table II.—Experiments with Water. Internal Diameter of Pipe, 3·30 cm.

v, mean velocity.	R.	$\frac{R}{\rho v^2} 10^2$.	$\frac{vd}{\nu} 10^{-3}$.	Temperature.
cm./sec.	dynes/cm.^{2}			
52·8	11·6	0·42	15·3	15° C.
71·3	18·7	0·37	20·7	
102·5	37·0	0·35	22·7	
151·0	73·1	0·32	43·7	
201·0	126·0	0·31	58·1	

Table III.—Experiments with Steam. Internal Diameter of Pipe, 1·07 cm.

Initial pressure.	v, mean initial velocity.	R.	$\frac{R}{\rho v^2} 10^2$.	$\frac{vd}{\nu} 10^{-3}$.
kgrm./cm.^{2}	cm./sec.	dynes/cm.^{2}		
11·8	2110	79·8	0·30	80·1
11·8	1530	46·3	0·33	58·1
11·6	2200	82·4	0·29	82·8
11·6	1940	67·8	0·31	73·1
11·6	936	17·9	0·35	35·2
5·06	2750	67·3	0·33	51·6
5·06	1310	17·1	0·37	24·7
5·06	3240	85·0	0·30	61·2
1·72	4320	63·9	0·35	32·6
1·76	2620	25·8	0·38	20·0
0·844	5480	54·0	0·36	22·6
0·844	5470	55·4	0·37	22·4

Table IV.—Experiments with Steam. Internal Diameter of Pipe, 1·905 cm.

Initial pressure.	v, mean initial velocity.	R.	$\frac{R}{\rho v^2} 10^2$.	$\frac{vd}{\nu} 10^{-3}$.
kgrm./cm.²	cm./sec.	dynes/cm.²		
15·12	6320	756·0	0·25	523·0
11·6	6360	590·0	0·25	426·0
11·6	6360	593·0	0·25	426·0
11·6	6090	540·0	0·25	409·0
11·6	5870	487·0	0·24	394·0
11·6	5360	420·0	0·25	359·0
11·6	4750	319·0	0·24	318·0
*15·12	2650	140·0	0·26	220·0
15·12	2190	95·3	0·26	182·0
15·12	938	19·4	0·29	78·0
11·6	3130	143·0	0·25	210·0
11·6	2370	83·1	0·25	159·0
*11·6	1820	52·8	0·27	122·0
11·6	1270	29·7	0·32	85·1
11·6	920	15·2	0·31	61·7
6·61	3920	149·0	0·28	164·0
6·61	3270	101·0	0·27	137·0
6·61	2820	80·6	0·29	118·0
6·61	2760	76·1	0·29	115·0
*6·61	2290	50·8	0·28	95·6
6·61	1550	25·0	0·30	648·0
4·57	4570	139·0	0·27	141·0
4·57	4360	130·0	0·28	134·0
4·57	3880	103·0	0·28	119·0
4·57	3240	76·6	0·29	99·5
*4·57	2780	56·8	0·30	85·4
4·57	2060	32·3	0·31	63·2
4·57	1230	12·6	0·34	37·8
2·81	5350	121·0	0·27	109·0
2·81	3240	52·2	0·32	66·0
2·81	1940	19·6	0·34	39·5
0·656	5200	36·1	0·34	31·0
0·656	4470	28·0	0·35	26·7
0·656	3940	22·6	0·37	23·5
*0·656	3170	15·4	0·39	18·9
0·656	3340	16·4	0·37	19·9
0·656	2940	13·7	0·40	17·5
0·656	2920	12·7	0·38	17·4

* These results are worked out in detail in Table VI.

Table V.—Experiments with Steam. Internal Diameter of Pipe, 3·30 cm.

Initial pressure.	v, mean initial velocity.	R.	$\frac{R}{\rho v^2} 10^2$.	$\frac{vd}{\nu} 10^{-3}$.
kgrm./cm.²	cm./sec.	dynes/cm.²		
15·1	1380	37·1	0·26	199·0
15·1	1700	56·4	0·28	244·0
15·1	2480	115·5	0·25	358·0
7·46	1340	20·2	0·29	107·0
7·46	1625	28·8	0·28	130·0
7·46	2240	53·5	0·28	178·0
7·46	2810	83·1	0·27	224·0
7·46	3170	104·0	0·27	252·0
3·72	704	34·4	0·35	31·4
3·72	2240	28·2	0·28	100·0
3·72	2820	43·2	0·27	126·0

Table VI.—Details of Reduction of Typical Results.

Internal Diameter of Pipe, 0·75 in. = 1·905 cm. Length of Pipe, 75·6 feet.

Initial pressure, p.	Initial pressure.	Pressure drop.	Average pressure drop per foot run.	Final pressure $= p_1 - \text{drop} = p^2$.	$p_1^{1\cdot94} - p_2^{1\cdot94}$.	Weight of steam entering,* w_0.	Weight of steam condensed in pipe,† λl.	$1 - \frac{\lambda l}{2w_0}$.
lb./in.^{2}	kgrm./cm.^{2}	lb./in.^{2}	lb./in.^{2}	lb./in.^{2}		lb./min.	lb./min.	
215·0	15·12	10·0	0·132	205·0	2955·0	7·53	0·104	0·993
165·0	11·6	3·70	0·0489	161·3	863·0	4·03	0·095	0·988
94·0	6·61	3·61	0·0478	90·39	491·0	3·00	0·074	0·988
65·0	4·57	4·04	0·0534	60·96	385·0	2·57	0·064	0·988
9·34	0·656	1·12	0·0148	8·22	16·73	0·473	0·026	0·972

$\frac{p_1^{1\cdot94} - p_2^{1\cdot94}}{1 - \lambda l/2w_0}\,\frac{1}{l}$ $= p_1^{1\cdot94} - p_3^{1\cdot94}$.	Pressure drop over first foot of pipe $= (p_1^{1\cdot94} - p_2^{1\cdot94}) \times 1/1{\cdot}94\,p_1^{\cdot94}$.	$R = \frac{\delta p \times D}{4}$.	$\rho \times 10^3$.	$v = \frac{w}{\rho \times a}$, mean initial velocity.	$\mu \times 10^6$.	$\nu \times 10^3$.	$\frac{R}{\rho v^2}\,10^2$.	$\frac{vd}{\nu}\,10^{-3}$.
	lb./in.^{2}	dynes/cm.^{2}	grm./cm.^{3}	cm./sec.				
39·3	0·130	140	7·53	2650	173	22·95	0·26	220·0
11·54	0·049	52·8	5·87	1820	167	28·4	0·27	122·0
6·56	0·0472	50·8	3·47	2290	158	45·6	0·28	95·6
5·14	0·0523	56·8	2·45	2780	152	62·0	0·30	85·4
2·27	0·0143	15·4	0·395	3170	126	319·0	0·39	18·9

* w_0 = water collected in hot well + water collected from low-pressure gauge.

† Weight of steam condensed in pipe was inferred from final dryness.

The above examples correspond respectively with the results marked with an asterisk (*) in Table IV.

Comparison of the Results with those of other Experimenters on Water and Air.

In fig. 3 the values of all the experimental results have been plotted with $R/\rho v^2$ as ordinates and $\log vd/\nu$ as abscissæ. The mean results as calculated from the formulæ of Ledoux for their steam velocity are also plotted on the diagram.

The boundary curves for Stanton and Pannell's results upon a drawn brass pipe are shown, together with their interpretation of Reynolds' experiments upon lead pipes. A curve has also been drawn illustrating Stanton's* experiments on artificially roughened pipes, in which by cutting screw threads in alternate directions on the inside of the pipe he succeeded in obtaining two pipes in which the resistance varied directly as the velocity squared. The resistance for these pipes as given by Stanton is expressed by the formula

$$R = 4{\cdot}6\, v_c^2 \times 10^{-6},$$

where v_c is the velocity at the axis. From the curve in Stanton's paper giving the radial distribution of velocity in these pipes the value of the mean velocity has been determined, the horizontal line in fig. 3 representing the resistance in the screwed pipes reduced to terms of the mean velocity.

It will be noted that the author's points for steam lie with fair accuracy upon a smooth curve slightly higher than Stanton and Pannell's boundary curves for low values of vd/ν becoming proportionately higher for large values. This characteristic is quite in conformity with the curves for rough and smooth brass pipes, the rough pipe offering proportionately much more resistance than the smooth pipe at the high values of vd/ν.

It has been shown by Lees† that the results of Stanton and Pannell's experiments can be represented with considerable accuracy by an expression of form

$$R = \rho v^2 \{a + b(\nu/vd)^n\},$$

the constants for air and water in smooth brass pipes being $a = 0{\cdot}0009$, $b = 0{\cdot}0763$, and $n = 0{\cdot}35$. With a modification of the constants the same expression represents the mean of the present experiments, the line in fig. 3 having been plotted using the following values: $a = 0{\cdot}002$, $b = 0{\cdot}141$, $n = 0{\cdot}44$.

From the general trend of their curves for high values of vd/ν Stanton and Pannell predicted a limit beyond which the resistance would be constant and hence independent of vd/ν. This is equivalent to the resistance varying as the square of the velocity. In the steam curves, owing probably to the

* 'Roy. Soc. Proc.,' A, vol. 85, p. 371.

† 'Roy. Soc. Proc.,' A, vol. 91, p. 46.

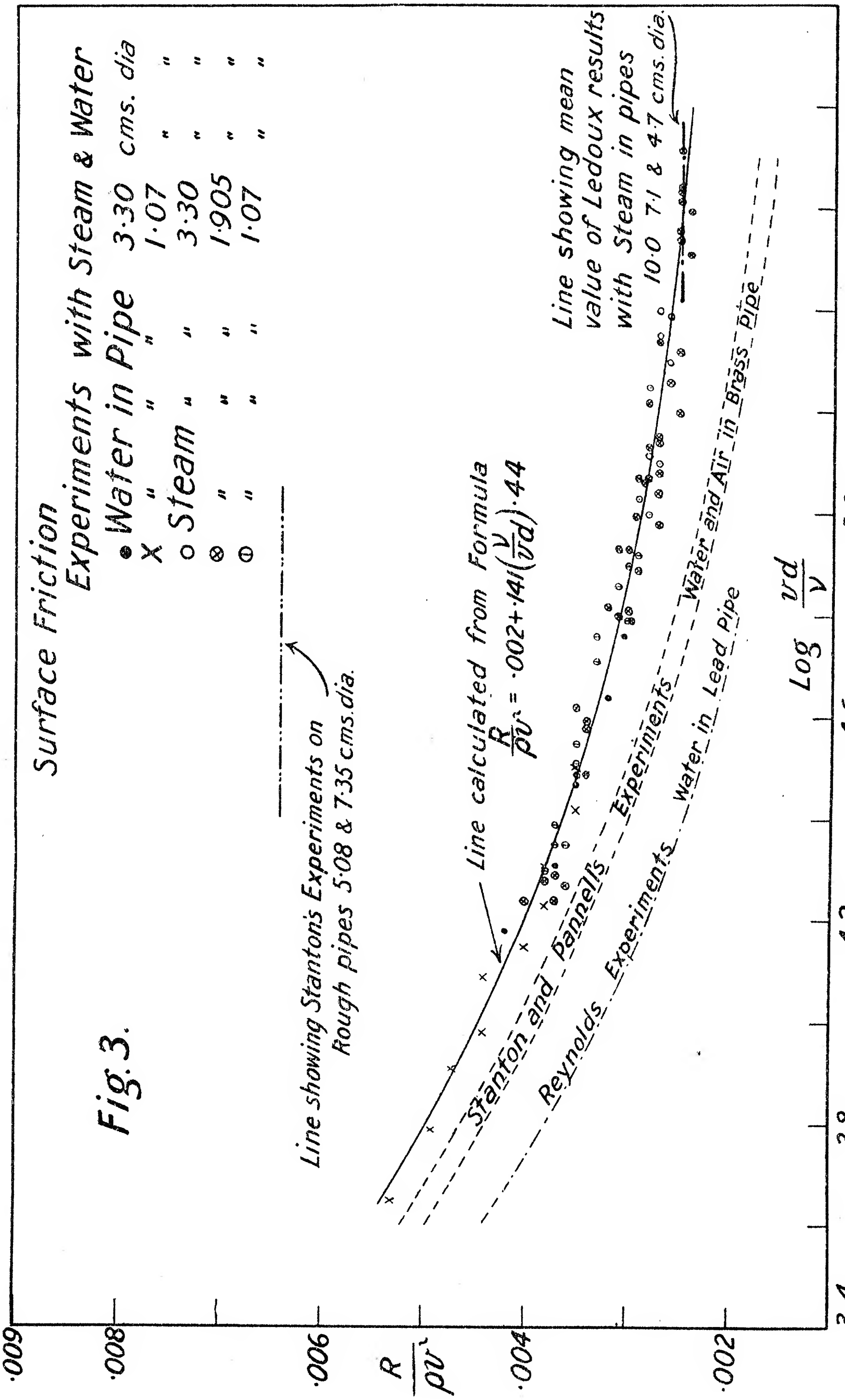
Surface Friction
Experiments with Steam & Water
● Water in Pipe 3·30 cms. dia
× " " " 1·07 " "
○ Steam " " 3·30 " "
⊗ " " " 1·905 " "
⊖ " " " 1·07 " "
Line showing Stanton's Experiments on Rough pipes 5·08 & 7·35 cms. dia.
Line calculated from Formula $\frac{R}{\rho v^2} = \cdot 002 + \cdot 141\left(\frac{\nu}{vd}\right)^{\cdot 44}$
Line showing mean value of Ledoux results with Steam in pipes 10·0 7·1 & 4·7 cms. dia.
Stanton and Pannells Experiments Water and Air in Brass Pipe
Reynolds Experiments Water in Lead Pipe
$\frac{R}{\rho v^2}$
·009
·008
·006
·004
·002
3·4
3·8
4·2
4·6
5·0
5·4
5·8
Log $\frac{vd}{\nu}$

Fig. 3.

pipes being slightly rougher, this limit would appear to be reached earlier and the ultimate value of $R/\rho v^2$ to be considerably above that corresponding to the smooth pipes used by Stanton and Pannell. The characteristics of the various curves appear to denote that the limiting value of vd/ν beyond which the resistance varies as the velocity squared becomes less with the roughness of the pipe surface, until, for a very rough surface, the resistance varies as the velocity squared for values only exceeding by small amounts that corresponding to the critical velocity.

The roughness of the steam pipe surface appears to be about the same as that of Darcy's pipes, the bulk of which were bitumen covered.

Rayleigh points out that if the law of similarity is to apply it must extend even to the proportionality of the slight projections and hollows which constitute the roughness of two given surfaces. Since in the present experiments three different diameters of pipe were used, the surfaces of which were probably of the same degree of roughness, it appears that, as was observed by Stanton and Pannell, in this region the effect of any want of proportionality of roughness is small.

The close agreement between the results obtained by steam and water in the same mains will also be noted.

Conclusions.

Further direct evidence has been produced to demonstrate the truth of the dimensional law, which is now shown experimentally to extend to the case of saturated vapours.

The fact that for the first time a range of pressures extending from about 15 inches of vacuum to 200 lb. per square inch above atmosphere has been used for the gas experiments is also not without interest. It is hoped that the work will prove of value to those who require data as to the resistance in steam and other mains, and who up to the present had to content themselves with a velocity squared law involving constants for comparatively few values of the velocity.

Again, the further demonstration of law of dynamical similarity appears to open up a wide field for testing purposes. The method of inferring the resistance of bodies of various shapes to the flow of air from experiments made on models in water has been in use for some time. The present work, by demonstrating the truth of the law as applied to the flow of steam and water in pipes, opens up the possibility of inferring the resistance of complicated steam passages, such as are met with in turbines, by tests on models according to the law of similarity.

It is hoped in the future to extend the work on these lines.

In conclusion the author desires to express his sincere thanks to Prof. J E. Petavel and the authorities of the University of Manchester for the valuable facilities granted for the carrying out of the work, and also to express his gratitude to the Council of the Royal Society for a monetary grant to defray the cost of the special apparatus needed.

Note on the Existence of Converging Sequences in Certain Oscillating Successions of Functions.

By Prof. W. H. Young, Sc.D., F.R.S.

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1. The object of the following note is to prove that, in a large class of important cases, a succession of functions can be shown to contain sub-sets of functions which converge. The interest attaching to the question is well known, and is sufficiently illustrated by the use I have made of these considerations, for example, in my paper† on the conditions that a trigonometrical series should have a certain form, as well as elsewhere. The first theorem on the subject is, as is there pointed out, due to Arzelà.

2. Theorem 1.—*Given a function which is upper semi-continuous on the left and lower semi-continuous on the right, there is a countable set of points dense everywhere, such that the value of the function at any point not belonging to the set is the unique limit of the values of the function at points of the set in a neighbourhood of the point when that neighbourhood shrinks up to the point.*

In fact, since the limits of approach are the same on the left and on the right, except at a countable set of points, it follows that, except at a countable set of points, the function is both lower semi-continuous and upper semi-continuous; in other words, that it is continuous, except at the points of such a set. Add to this set, should it not be everywhere dense, any countable everywhere dense set, and we get such a set S as that contemplated in the enunciation. For a point not belonging to it the value is the unique limit of values in the neighbourhood equally, whether we confine our attention to points on the countable set S or not.

* A revision of one part of the proofs, due to Mrs. Grace Chisholm Young, was received on October 25, 1915, and has been incorporated in the paper.

† Published in these 'Proceedings,' vol. 88. The theorem contained in Cor. to Theorem 4 of para. 5 of the present communication is there utilised, but its proof is in part based on an erroneous theorem, quoted from the 'Proc. London Math. Soc.'